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SHAPE OPTIMIZATION OF TWO-DIMENSIONAL THERMAL CONDUCTING SOLID USING BOUNDARY INTEGRAL EQUATION FORMULATION

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A method for shape design sensitivity analysis of two-dimensional thermal conducting solid is presented using the material derivative concept and the adjoint variable method. A general thermal boundary condition with heat convection is considered in addition to prescribed temperature and heat flux. The method for deriving the sensitivity formula is based on standard direct boundary integral equation formulation. The sensitivity of a general functional depending on temperature and heat flux is considered. The method is then applied to obtain the sensitivity formulas for temperature and heat flux constraints imposed over a small segment of the boundary. The accuracy of the sensitivity analysis is demonstrated by a hollow cylinder problem with exact solution. A weight minimization problem of a thermal diffuser is considered as a practical application. The sensitivity by the presented method is compared with that by finite differences and an optimal shape is found by use of an optimization routine.

Key Words: Shape Optimization, Sensitivity Analysis, Boundary Integral Equation, Boundary Element Method, Thermal Conduction, Thermal Diffuser

1. INTRODUCTION

The accurate calculation of a shape design sensitivity can be a crucial ingredient for successful shape optimization algorithm. Since the boundary element method (BEM) has emerged as an alternative to the finite element method (FEM), it has naturally been accepted as an efficient and attractive tool in the shape optimization field because of its relatively reduced dimensionality, accuracy of the boundary solution and convenience in regridding. Many researchers applied the BEM to shape optimization problem (see for survey, Mota Soares et al., 1987). Kane et al. (1988, 1991) and Saigal et al. (1989) have presented implicit differentiation method for the shape design sensitivity analysis (SDSA) based on the BEM, where the boundary integral equation (BIE) is first discretized, and the derivative of the system matrix is calculated analytically. On the other hand, there have been continued research works for the continuum approach of SDSA based on the boundary integral formulation. In the continuum approach, the shape sensitivity formula is first derived analytically based on the continuum basis, and next discretized for the calculation of the state variables and sensitivity. Kwak and Choi (1987), Choi and Kwak (1988 a, 1988 b), Kwak and Lee (1990), and Lee and Kwak (1991, 1992) have presented a general procedure for the adjoint method of the SDSA using the formal BIE formulation, where the adjoint systems are introduced to derive the sensitivity formula of the general performance functional. Barone and Yang (1988, 1989), Rice and Mukherjee (1990), Choi and Choi (1990), and Choi and Kwak (1990) have used the direct differentiation method, where the sensitivity of state variables is determined by solving a new BIE established by differentiation of the system equation.

There have been, however, few works dealing with the shape optimization of thermal conducting solid in spite of its significance. Delfour et al. (1983) have optimized the shape of a thermal diffuser to achieve minimum weight by use of the FEM. Dems (1987) and Tortorelli et al. (1989) have presented the shape sensitivity formulation by use of Lagrangian multiplier technique to define the adjoint system. They also used the FEM for the solution of system equations. Meric (1988) has derived a shape sensitivity formula for nonlinear anisotropic themal problem using Lagrangian multiplier technique. Although the whole process of the derivation was independent of the BEM, he has proposed to use the BEM because of its accuracy of boundary solution. Park and Yoo (1988) have derived a shape sensitivity formula for thermal problem based on the variational formulation using the material derivative concept (Haug et al., 1986) and the adjoint variable method. They have transformed the variational equation defining the primary and adjoint systems to a BIE.

In this paper, a shape sensitivity formula of general performance functional depending on temperature and heat flux is derived for a two-dimensional thermal conducting solid. The method directly extends that of Kwak and Choi (1987), which has covered a potential problem, to a thermal conduction problem. While Kwak and Choi have dealt with boundary condition with prescribed potential and flux only, more general thermal boundary condition is considered in this paper, that is, heat convection boundary condition is included in addition to the prescribed temperature and heat flux. The

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direct differentiation method (Choi and Choi, 1990, Choi and Kwak, 1990), having its own advantage, can be utilized for deriving the formula, but the method here is based on the standard direct BIE formulation, and uses the material derivative concept and the adjoint variable method. It is then applied to obtain the sensitivity formulas for temperature and heat flux constraints on the boundary. The formulas are derived for the shape variations in the normal and tangential directions. It is noted that while the previous sensitivity formulas by Park and Yoo(1988) and Meric(1988) are derived from variational formulation and Lagrangian multiplier method, respectively, ours are derived based purely on BIE formulation. Accuracy of the sensitivity calculation is demonstrated through an example of a hollow cylinder with analytic solution. For a two-dimensional thermal diffuser problem, the sensitivity calculated by the present method is compared with that by finite differences. Finally, an optimal shape to minimize the wieght of a thermal diffuser is found by incorporation of the sensitivity calculation algorithm in an optimization routine. It is shown that the inclusion of tangential component of the design velocity can be very useful when the boundary variation is constrained in some directions as the present example.

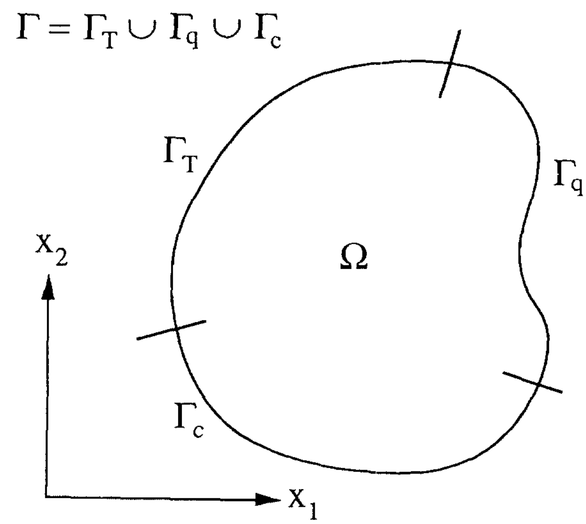

Fig. 1 A two-dimensional thermal conducting solid

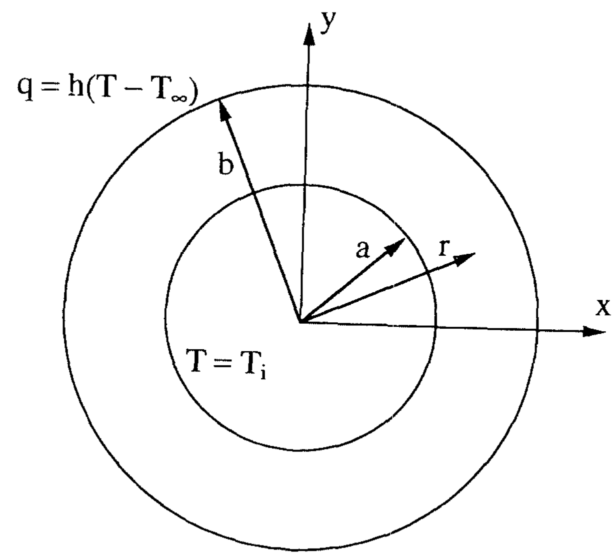

Fig. 2 A hollow cylinder problem

2. BIE FORMULATION FOR TWO-DIMENSIONAL THERMAL CONDUCTING SOLID

Consider a two-dimensional steady-state thermal conduction problem for an isotropic and homogeneous solid body Ω of arbitrary shape with a sufficiently smooth boundary Γ as shown in Fig. 1. The position in the solid body will be denoted by x or x_o when necessary. If the temperature is denoted by $T(x)$, then the heat flux vector representing heat flow rate per unit area is expressed as

$$q_i(T) = -kT_{,i} \tag{1}$$

where k is the thermal conductivity of the solid. The indices in the tensor notations have a range of 1 and 2. Then the equilibrium equation is given by

$$-q_{i,i}(T) + Q(x) = 0, \quad x \in \Omega \tag{2}$$

where $Q(x)$ is the prescribed internal heat generation rate per unit volume. A general thermal boundary condition is applied on the boundary $\Gamma = \Gamma_T \cup \Gamma_q \cup \Gamma_c$ as

$$\begin{aligned} &T(x) = T_o(x) && x \in \Gamma_T \\ &q(T) \equiv q_i(T)\, n_i = q_o(x) && x \in \Gamma_q \\ &q(T) \equiv q_i(T)\, n_i = h\{T(x) - T_\infty(x)\} && x \in \Gamma_c \end{aligned} \tag{3}$$

where q is the normal heat flux, n_i the outward unit normal vector on the boundary. Temperature $T_o(x)$ is prescribed on Γ_T, normal heat flux $q_o(x)$ on Γ_q, and constant heat transfer coefficient h and surrounding temperature $T_\infty(x)$ on Γ_c.

Consider a general performance functional composed of domain and boundary integrals arising in the shape optimization of the thermal conducting solid in the following form

$$\Phi = \int_\Omega \xi(T, q_i(T))\, dx + \int_\Gamma \Psi(T, q(T))\, ds \tag{4}$$

where ξ and Ψ are smooth functions which are continuously differentiable with respect to their arguments, and ds and dx represent integration with respect to x along Γ and in Ω. It is now desired to derive an explicit expression for the variation of Φ in terms of boundary movements.

By applying Green's formula to the thermal conduction problem, the following direct BIE is obained (Brebbia et al., 1984, Banerjee et al., 1981)

$$\begin{aligned} c(x_o)\, T(x_o) = &\int_\Gamma \{-q(x)\, G(x_o, x) + T(x)\, F(x_o, x)\} ds \\ &+ \int_\Omega Q(x)\, G(x_o, x)\, dx, \quad x_o \in \Omega \cup \Gamma \end{aligned} \tag{5}$$

where c is a constant determined by the position of x_o. The kernels $G(x_o, x)$ and $F(x_o, x)$ in Eq. (5) are Green's fundamental solutions given by

$$\begin{aligned} G(x_o, x) &= \frac{1}{2\pi k} \ln\left(\frac{1}{r}\right) \\ F(x_o, x) &= \frac{n_i r_i}{2\pi r^2} \end{aligned} \tag{6}$$

where r is the distance between the load point x_o and the field point x, and the following relation holds.

$$\begin{aligned} r^2 &= r_i r_i \\ r_i &= x_i - x_{oi} \end{aligned} \tag{7}$$

For x_o on Γ, Eq. (5) can now be transformed into the following equation by multiplying an arbitrary function ρ^* defined on Γ, and integrating over Γ.

$$\int_\Gamma T(x)\left\{c(x)\rho^*(x)-\int_\Gamma \rho^*(x_o)F(x_o,x)\,ds_o\right\}ds+\int_\Gamma q(x)\int_\Gamma \rho^*(x_o)G(x_o,x)\,ds_o ds$$
$$=\int_\Omega Q(x)\int_\Gamma \rho^*(x_o)G(x_o,x)\,ds_o dx \tag{8}$$

where ds_o represents integration with respect to x_o along Γ. We can consider, on the other hand, an arbitrary system of temperature T^* and normal heat flux q^*, which can be expressed by the indirect BIE formulation (Banerjee et al., 1981) as

$$T^*(x)=\int_\Gamma \rho^*(x_o)G(x_o,x)\,ds_o+\int_\Omega Q^*(x_o)G(x_o,x)\,dx_o,\quad x\in\Omega\cup\Gamma$$
$$q^*(T^*(x))=-c(x)\rho^*(x)+\int_\Gamma \rho^*(x_o)F(x_o,x)\,ds_o+\int_\Omega Q^*(x_o,x)\,dx_o,\quad x\in\Gamma \tag{9}$$

where ρ^* can be interpreted in this case as the source density distributed on Γ, and Q^* corresponds to another heat source density in Ω. Then T^*, q^* and Q^* satisfy the thermal equilibrium equation as

$$-q^*_{i,i}(T^*)+Q^*(x)=0,\quad x\in\Omega \tag{10}$$

Substituting Eq. (9) into (8) and using (5) for x_o in Ω, the following boundary integral identity is obiained

$$\int_\Gamma\{Tq^*(T^*)-q(T)T^*\}ds=\int_\Omega(TQ^*-QT^*)\,dx \tag{11}$$

This equation corresponds to Green's second identity which holds between two arbitrary equilibrium states : one with temperature T, heat flux q and heat source density Q, and the other with T^*, q^* and Q^*.

3. METHOD OF SDSA

A complete formula for the SDSA is now derived, based on the BIE formulation in the previous section, for the general performance functional given in Eq. (4). Since the shape change can be considered as a deforming body, the material derivative concept (Haug et al., 1986) is utilized. Then the variation of Φ is expressed by the material derivative formula, using the design velocity $V=\{V_1, V_2\}$, as

$$\Phi'=\int_\Omega(\dot{\xi}+\xi V_{k,k})\,dx+\int_\Gamma\{\dot{\Psi}+\Psi(V_{k,k}-V_{i,j}n_i n_j)\}ds \tag{12}$$

The velocity field on the boundary can be decomposed into the normal and tangential components such that

$$V_i=V_n n_i+V_s s_i \tag{13}$$
$$V_{k,k}-V_{i,j}n_i n_j=V_n H+V_{s,s} \text{ on } \Gamma \tag{14}$$

where H is the curvature of the boundary, s_i the unit tangential vector on the boundary, and subscript s after a comma denotes a derivative in the tangential direction on the boundary.

Expanding $\dot{\xi}$ and $\dot{\Psi}$ in terms of their respective arguments and performing integration by parts, Eq. (12) can be rewritten as

$$\Phi'=\int_\Omega(\xi_T-q^o_{i,i})\,\dot{T}dx-\int_\Omega\{\xi_T T_{,j}V_j+\xi_{q_i}q_i(T_{,j}V_j)\}dx$$
$$+\int_\Gamma\{(\Psi_T+q^o)\dot{T}+\Psi_q\dot{q}\}ds$$
$$+\int_\Gamma\{(\xi+\Psi H)V_n-\Psi_{,s}V_s\}ds$$
$$+\sum<\Psi V_s> \tag{15}$$

where the subscripts to ξ and Ψ denote partial derivatives and the following expression is introduced.

$$q_i^o\equiv-k\xi_{q_i},\ q^o\equiv q_i^o n_i \tag{16}$$

Note that $V_{s,s}$ in Eq. (14) has been removed in Eq. (15) by integration by parts on the boundary, resulting in the jump terms denoted by $\sum<.>$ at the point with discontinuity. Now, to represent $\dot{T}$ and $\dot{q}$ explicitly with design velocity V, material derivative is taken to the boundary integral identity (11). After some manipulations and simplifications, we obtain the final expression for the variation of Eq. (11) as

$$\int_\Omega Q^*\dot{T}dx+\int_\Gamma(\dot{q}T^*-\dot{T}q^*)\,ds$$
$$=\int_\Omega Q^*T_{,j}V_{,j}V_j dx$$
$$+\int_\Gamma[\{QT^*-kT_{,s}T^*_{,s}+q(q^*/k-T^*H)\}V_n$$
$$+(q_{,s}T^*-T_{,s}q^*)V_s]ds+\sum<-T^*qV_s> \tag{17}$$

As in Eq. (15), $V_{n,s}$ and $V_{s,s}$ have been removed in Eq. (17) by integration by parts, resulting in the jump terms. In order to relate $\dot{T}$ and $\dot{q}$ in Eq. (15) to those in (17), an adjoint system is introduced as follows.

The adjoint system is defined by the indirect BIE (9) with

$$Q^*=\xi_T-q^o_{i,i} \text{ in } \Omega \tag{18}$$

and the boundary conditions

$$\begin{aligned} T^*&=\Psi_q && \text{on } \Gamma_T\\ q^*&=-(\Psi_T+q^o) && \text{on } \Gamma_q\\ q^*&=h(T^*-\Psi_q)-(\Psi_T+q^o) && \text{on } \Gamma_c \end{aligned} \tag{19}$$

Substituting Eq. (3), (18) and (19) into Eqs (15) and (17) and performing integration by parts for the elimination of the domain integral, we finally obtain the sensitivity formula as follows

$$\Phi'=\int_{\Gamma T}(\Psi_T+q^o+q^*)\,\dot{T}_o ds+\int_{\Gamma q}(\Psi_q-T^*)\,\dot{q}_o ds$$
$$+\int_\Gamma[\{\xi+\Psi H+(Q-qH)T^*+q(q^*+q^o)/k-kT_{,s}T^*_{,s}\}V_n$$
$$+\{q_{,s}T^*-T_{,s}(q^*+q^o)-\Psi_{,s}\}V_s]ds+\sum<(-T^*q+\Psi)V_s> \tag{20}$$

Note that the adjoint system has the form of a conventional indirect BIE. However, it can be shown (Banerjee et al., 1981) that the direct BIE (5) and the indirect BIE (9) are equivalent. Since the adjoint system of BIE (9) has the same kernels as the primary BIE (5), if the adjoint system data, (18) and (19), are sufficiently smooth, there will be solutions as already studied in Park and Yoo (1988). In addition, the use of the direct BEM in the primary and adjoint systems can enhance efficiency of the numerical implementation. When the solutions of the primal and adjoint systems are obtained. Φ' can be calculated by Eq. (20) by the standard Gaussian integration procedure using the same discretization model as in the boundary element analysis.

4. SENSITIVITY FORMULAS FOR TEMPERATURE AND HEAT FLUX CONSTRAINTS

To show the application of the method of SDSA, sensitivity formulas are presented here for temperature and heat flux constraint functionals arising in the shape optimization problem. Because temperature and heat flux are critical on the boundary in many applications, the constraints are imposed over a small part Γ_i of the boundary as follows

$$\Phi_T=\frac{1}{L_i}\int_{\Gamma i}\left(\frac{T}{T_c}-1\right)ds\leq 0 \tag{21}$$

$$\Phi_q=\frac{1}{L_i}\int_{\Gamma i}\left(\frac{q}{q_c}-1\right)ds\leq 0 \tag{22}$$

Here T_c and q_c are constants defining the limits on temperature and heat flux on the constrained boundary, respectively, and L_i represents the length of the boundary sengment Γ_i as

$$L_i=\int_{\Gamma i}ds \tag{23}$$

Sensitivity formulas can be derived by following the same procedure described in the previous section, and are summarized as follows

$$\Phi'_T=\frac{1}{L_i}\int_{\Gamma i}\left(\frac{T}{T_c}-1-\Phi_T\right)(V_nH+V_{s,s})\,ds+F(V) \tag{24}$$

$$\Phi'_q=\frac{1}{L_i}\int_{\Gamma i}\left(\frac{q}{q_c}-1-\Phi_q\right)(V_nH+V_{s,s})\,ds+F(V) \tag{25}$$

where

$$\begin{aligned}F(V)=&\int_{\Gamma T}q^*\dot{T}_o ds-\int_{\Gamma q}T^*\dot{q}_o ds+\int_{\Gamma}[\{(Q-qH)\,T^*\\&+qq^*/k-kT_{,s}T_{,s}\}\,V_n+(q_{,s}T^*-T_{,s}q^*)\,V_s]\,ds\\&+\sum<-T^*qV_s>\end{aligned} \tag{26}$$

The condition for the adjoint system is determined depending on the kind of the primal boundary condition on the constrained boundary Γ_i as follows,

(1) When heat flux is prescribed on the temperature-constrained boundary ($\Phi=\Phi_T$, $\Gamma_i\subset\Gamma_q$)

$$\begin{aligned}&Q^*=0 && \text{in } \Omega\\&T^*=0 && \text{on } \Gamma_T\\&q^*=-\frac{1}{L_iT_c} && \text{on } \Gamma_i\\&q^*=0 && \text{on } \Gamma_q\backslash\Gamma_i\\&q^*=hT^* && \text{on } \Gamma_c\end{aligned} \tag{27}$$

(2) When heat convection condition is assigned on the temperature-constrained boundary ($\Phi=\Phi_T$, $\Gamma_i\subset\Gamma_c$)

$$\begin{aligned}&Q^*=0 && \text{in } \Omega\\&T^*=0 && \text{on } \Gamma_T\\&q^*=0 && \text{on } \Gamma_q\\&q^*=h\left(T^*-\frac{1}{hL_iT_c}\right) && \text{on } \Gamma_i\\&q^*=hT^* && \text{on } \Gamma_c\backslash\Gamma_i\end{aligned} \tag{28}$$

(3) When temperature is prescribed on the heat flux-constrained boundary ($\Phi=\Phi_q$, $\Gamma_i\subset\Gamma_T$)

$$\begin{aligned}&Q^*=0 && \text{in } \Omega\\&T^*=\frac{1}{L_iq_c} && \text{on } \Gamma_i\\&T^*=0 && \text{on } \Gamma_T\backslash\Gamma_i\\&q^*=0 && \text{on } \Gamma_q\\&q^*=hT^* && \text{on } \Gamma_c\end{aligned} \tag{29}$$

(4) When heat convection condition is assigned on the heat flux-constrained boundary ($\Phi=\Phi_q$, $\Gamma_i\subset\Gamma_c$)

$$\begin{aligned}&Q^*=0 && \text{in } \Omega\\&T^*=0 && \text{on } \Gamma_T\\&q^*=0 && \text{on } \Gamma_q\\&q^*=h\left(T^*-\frac{1}{L_iq_c}\right) && \text{on } \Gamma_i\\&q^*=hT^* && \text{on } \Gamma_c\backslash\Gamma_i\end{aligned} \tag{30}$$

5. NUMERICAL EXAMPLES

5.1 A Hollow Cylinder Problem

As an example to check the accuracy of the method of SDSA, a hollow cylinder with exact solution, as shown in Fig. 2, is considered here. Inner and outer radii of the circular cylinder are a and b, respectively. The inner surface is kept at constant temperature T_i. Heat convection occurs on the outer surface with surrounding medium at bulk temperature T_∞ and heat transfer coefficient h. The exact temperature distribution can be found in Reference (Carslaw et al., 1959). By taking the outer radius b as the design variable, we can obtain the exact sensitivities. Details of the analytical sensitivities are given in Appendix.

To obtain primal and adjoint solutions, a quadratic boundary element model is used. Due to symmetry of the problem, one eighth sector of the circular section is modelled as shown in Fig. 3. Temperature and heat flux functionals are defined in the average sense as

$$\begin{aligned}\Phi_T&=\frac{1}{L_i}\int_{\Gamma i}Tds\\\Phi_q&=\frac{1}{L_i}\int_{\Gamma i}qds\end{aligned} \tag{31}$$

where the subscript i represents the element number in Fig. 3. Γ_i corresponds to the boundary segment spanned by the element number i, and L_i is its length. The temperature functional is considered for both the heat flux and the heat convection prescribed boundaries, and the heat flux functional for the temperature and the heat convection prescribed boundaries. Because the design variable is the outer radius, the design velocity becomes

$$\begin{aligned}&V_n=\delta b \text{ and } V_s=0 && \text{on } \Gamma_{B1}\\&V_n=0 \text{ and } V_s=\frac{\rho-a}{b-a}\delta b && \text{on } \Gamma_{B4}\end{aligned}$$

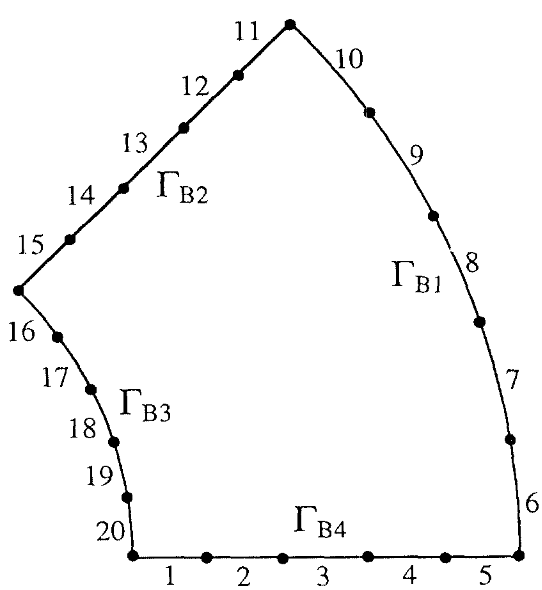

Fig. 3 A quadratic boundary element model and element numbers for the hollow cylinder problem

$$V_n=0 \quad \text{and} \quad V_s=-\frac{\rho-a}{b-a}\delta b \qquad \text{on } \Gamma_{B2} \tag{32}$$

where ρ represents the radius from center of the cylinder. If we use the material derivative formula as

$$\Phi'=\frac{d\Phi}{db}\delta b \tag{33}$$

then we can obtain the sensitivity expression as follows

$$\begin{aligned}
\frac{d\Phi_T}{db}&=\frac{1}{L_i}\int_{\Gamma i}\frac{T-\Phi_T}{b}ds+C && \text{if } \Gamma_i\subset\Gamma_{B1}\\
\frac{d\Phi_T}{db}&=\frac{1}{L_i}\int_{\Gamma i}\frac{T-\Phi_T}{b-a}ds+C && \text{if } \Gamma_i\subset\Gamma_{B2}\\
\frac{d\Phi_q}{db}&=\frac{1}{L_i}\int_{\Gamma i}\frac{q-\Phi_q}{b}ds+C && \text{if } \Gamma_i\subset\Gamma_{B1}\\
\frac{d\Phi_q}{db}&=C && \text{if } \Gamma_i\subset\Gamma_{B3}
\end{aligned} \tag{34}$$

where

$$C=\int_{\Gamma B1}\left\{-kT_{,s}T^*_{,s}+q\left(\frac{q^*}{k}-\frac{T^*}{b}\right)\right\}ds-\int_{\Gamma B2}(q_{,s}T^*-T_{,s}q^*)\frac{\rho-a}{b-a}ds+\int_{\Gamma B4}(q_{,s}T^*-T_{,s}q^*)\frac{\rho-a}{b-a}ds \tag{35}$$

The adjoint systems are defined by the Eqs. (27) to (30), where T_c and q_c should be replaced by 1.

For numerical calculation, a, b, T_i, T_∞, k and h are taken as 0.1 m, 0.2 m, 100℃, 0℃, 47 W/m℃, and 10 W/m^2℃, respectively. Calculated temperature and heat flux sensitivities are listed in Table 1 and 2, respectively, and are compared with analytic sensitivities at mid points of the boundary elements. Very accurate results are obtained, as can be seen in Table 1 and 2. The ratio in Table 1 and 2 is calculated by dividing the calculated sensitivity by the analytic sensitivity. The maximum percentage errors are 2.44 in the temperature sensitivity and 1.74 in the heat flux sensitivity.

Table 1 Temperature sensitivities for the hollow cylinder problem

Element number	Exact sensitivity	SDSA result	Ratio(%)
1	−5.58956	−5.45306	97.56
2	−14.58143	−14.50218	99.46
3	−21.57130	−21.51642	99.75
4	−27.21890	−27.17696	99.85
5	−31.91611	−31.89470	99.93
6	−33.98974	−33.99204	100.01
7	−33.98974	−33.98730	99.99
8	−33.98974	−33.98861	100.00
9	−33.98974	−33.98730	99.99
10	−33.98974	−33.99204	100.01

Table 2 Heat flux sensitivities for the hollow cylinder problem

Element number	Exact sensitivity	SDSA result	Ratio(%)
6	−339.897	−339.920	100.00
7	−339.897	−339.873	100.00
8	−339.897	−339.886	100.00
9	−339.897	−339.873	100.00
10	−339.897	−339.920	100.00
16	−9033.700	−9191.208	101.74
17	−9033.700	−9016.260	99.81
18	−9033.700	−9032.436	99.99
19	−9033.700	−9016.260	99.81
20	−9033.700	−9191.208	101.74

5.2 A Thermal Diffuser Problem

A weight minimization problem of a two-dimensional thermal diffuser is considered here as an application of the shape optimization. The problem is similar to that treated by Delfour et al. (1983). Only one half of the thermal diffuser is considered because of symmetry, as shown in Fig. 4. The diffuser has a uniform inward heat flux of 1000 W/m^2 at the heat source surface Γ_2 and a fixed temperature of 0℃ at the heat discharge surface Γ_4. The lateral surface Γ_1 is kept insulated. The problem is to find the best shape of the diffuser to minimize the weight such that the outward heat flux at Γ_4 does not exceed the prescribed limit q_c. The value of q_c is set as 300 W/m^2. The boundary segment Γ_1 is to be varied, while end points P_1 and P_2 are fixed.

A boundary element model with 40 quadratic elements is used, as shown in Fig. 5. The design boundary is represented by a cubic spline function with relaxed end conditions at points P_1 and P_2, and the y coordinates of nodes 1 to 14 in Fig. 5 are selected as the design variables to generate the spline. Then, the design velocity is expressed as

$$V_n=n_y\delta y \qquad V_s=n_x\delta y \tag{36}$$

Heat flux constraints defined by Eq. (22) are imposed over all elements on boundary Γ_4, generating 10 constraints. Because a temperature boundary condition is prescribed on

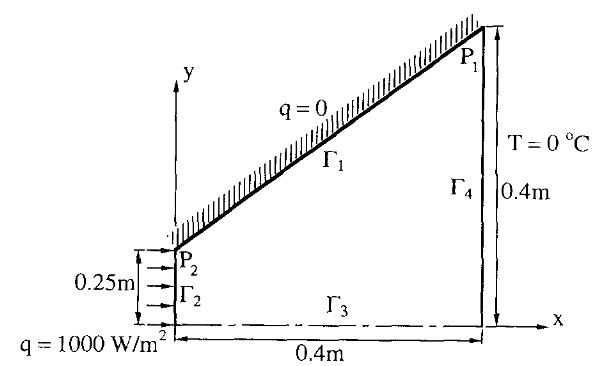

Fig. 4 A two-dimensional thermal diffuser problem

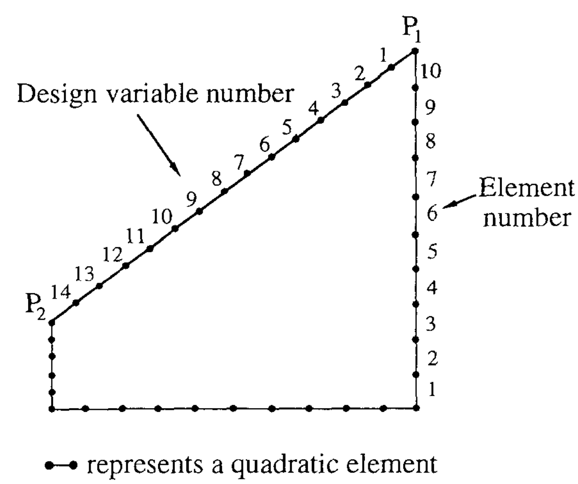

Fig. 5 A quadratic boundary element model at the initial design for the thermal diffuser problem

Table 3 Heat flux sensitivities for the thermal diffuser problem

Element number	$\Phi_q^{(i)}$	$\Phi_q^{(m)}$	$\Delta\Phi_q$	Φ_q'	$\Phi_q'/\Delta\Phi_q\times 100$
1	0.2176	0.2169	−0.7270E−03	−0.7284E−03	100.19
2	0.1914	0.1907	−0.6999E−03	−0.7006E−03	100.10
3	0.1402	0.1395	−0.6447E−03	−0.6459E−03	100.18
4	0.0658	0.0651	−0.5595E−03	−0.5605E−03	100.17
5	−0.0294	−0.0298	−0.4401E−03	−0.4408E−03	100.15
6	−0.1430	−0.1433	−0.2786E−03	−0.2788E−03	100.09
7	−0.2739	−0.2739	−0.5795E−04	−0.5769E−04	99.56
8	−0.4227	−0.4224	0.2623E−03	0.2630E−03	100.25
9	−0.5952	−0.5943	0.8314E−03	0.8493E−03	102.22
10	−0.8174	−0.8151	0.2316E−02	0.2300E−02	99.29

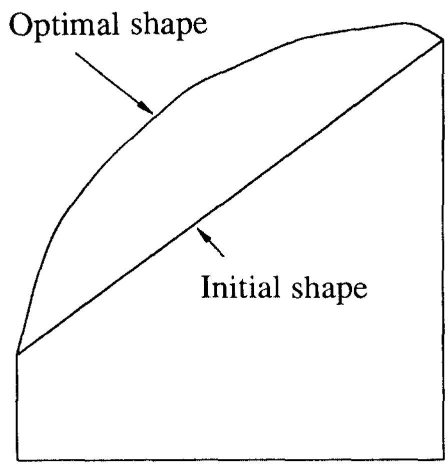

Fig. 6 Optimal shape of the thermal diffuser

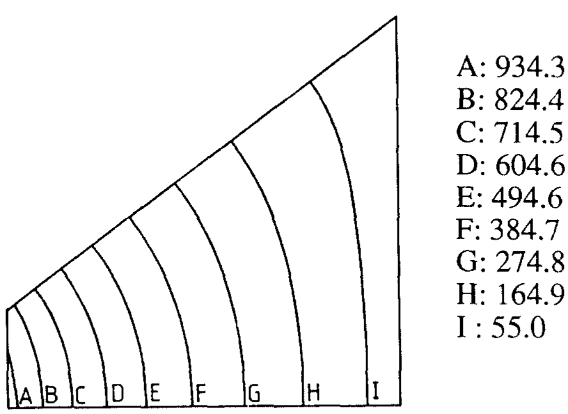

Fig. 7 Temperature distribution of the thermal diffuser at the initial design

the heat flux-constrained boundary, the adjoint system is defined by Eq. (29). The sensitivity expression is obtained as follows

$$\Phi'_q=-\int_{\Gamma_1} T_{,s}(kT^*_{,s}n_y+q^*n_x)\,\delta y ds \tag{37}$$

In this example, the sensitivity calculated by the present method is compared with that by finite differences. A uniform design change (0.1%) of δy, i.e., $\delta y=0.001y$ is used in the finite differencing, which is small enough to predict the actual

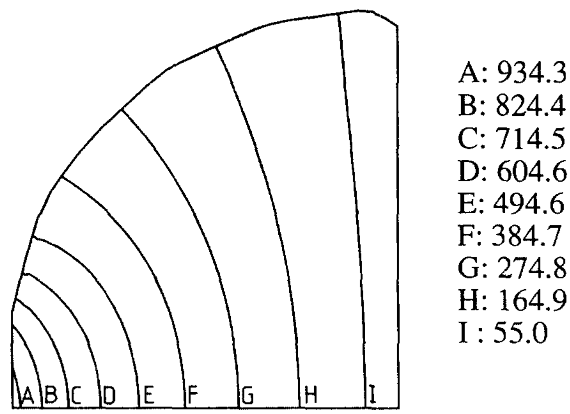

Fig. 8 Temperature distribution of the thermal diffuser at the optimal design

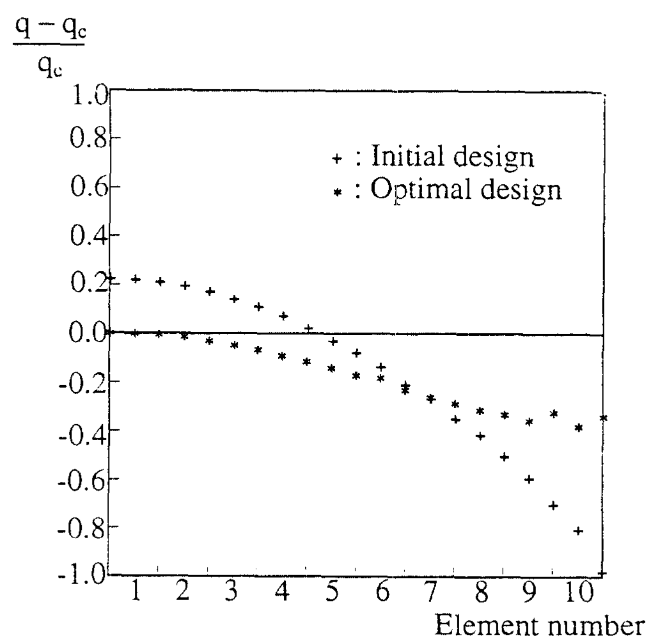

Fig. 9 Heat flux distribution along the heat flux-constrained boundary

sensitivity. Define $\Phi_q^{(i)}$ and $\Phi_q^{(m)}$ as the functional values for the initial and perturbed designs, respectively, and let $\Delta\Phi_q$ be $\Phi_q^{(m)}-\Phi_q^{(i)}$ and $\Phi_q^{(i)}$ and Φ_q' be the predicted difference values by the SDSA. Results of the heat flux sensitivities are listed in Table 3. Fairly accurate results are obtained. The maximum percentage error is 2.22.

The presented method of SDSA is incorporated in the optimization routine IDESIGN (Arora, 1984). The acceptable constraint violation at the optimum is set as 0.1 per cent of the prescribed upper limit of the outward heat flux. The optimal design is obtained in 25 iterations and is shown in Fig. 6. The weight at the optimal design is 1.374, which is the value normalized by the weight at the initial design. The temperature distribution at the initial and optimal designs are shown in Fig. 7 and 8, respectively. Fig. 9 shows the heat flux distributions along the constrained boundary Γ_4, where the amount of deviation from q_c has decreased noticeably from the initial design to the optimal design.

6. CONCLUSIONS

A generalized shape design sensitivity formula has been derived for two-dimensional thermal conducting solid. General thermal boundary conditions are considered, including heat convection in addition to prescribed temperature and heat flux. The method for deriving the formula uses the material derivative concept and the adjoint variable lethod, and is based on standard direct BIE formulation. The method is illustrated by derivng explicit formulas for temperature and heat flux constraint functionals defined on the boundary. The accuracy has been checked by a hollow cylinder problem with an exact sensitivity. As a practical example, a thermal diffuser problem with heat flux constraint on the heat discharge surface is dealt with. For the problem, the calculated sensitivity is compared with that by finite differences. An optimal shape to minimize the weight is found by use of an optimization program. An extension to axisymmetric and three-dimensional cases can be done by following the same procedure. An implementation to axisymmetric thermal conducting solid is under study.

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APPENDIX

The analytical expressions for the temperature and heat flux sensitivities of the hollow cylinder problem are summarized here. The temperature distribution as a function of radius ρ from center of the cylinder can be expressed as (Carslaw et al., 1959)

$$T=[T_i\{1+h'b\ \ln(b/\rho)\}+T_\infty h'b\ \ln(\rho/a)]/\{1+h'b\ \ln(b/a)\} \tag{A1}$$

where h' represents h/k, h and k being heat transfer coefficient and thermal conductivity, respectively. The outward

$$q|_{\rho=a}=-hb(T_i-T_\infty)/[a\{1+h'b\ \ln(b/a)\}]$$
$$q|_{\rho=b}=h(T_i-T_\infty)/\{1+h'b\ \ln(b/a)\} \tag{A2}$$

The temperature sensitivity as a function of radius ρ can be obtained by differentiating equation $(A1)$ as

$$\frac{dT}{db}=[\{1+h'b\ln(b/a)\}\{T_i+T_i\ln(b/\rho)+T_\infty\ln(\rho/a)\} -\{1+\ln(b/a)\}\{T_i+T_ih'b\ \ln(b/\rho)+ T_\infty h'b\ \ln(\rho/a)\}]\,h'/ \{1+h'b\ \ln(b/a)\}^2+(-T_i+T_\infty)h'b(1-a/\rho)/ [(b-a)\{1+h'b\ \ln(b/a)\}] \tag{A3}$$

The heat flux sensitivities on the inner and outer surfaces can be obtained by differentiating equation (A2) as follows

$$\frac{d}{db}(q|_{\rho=a})=-(T_i-T_\infty)kh'^2\{1+\ln(b/a)\}/\{1+h'b\ln(b/a)\}^2$$
$$\frac{d}{db}(q|_{\rho=b})=-(T_i-T_\infty)kh'(1-h'b)/[a\{1+h'b\ln(b/a)\}^2] \tag{A4}$$